DIOPHANTINE ANALYSIS.

101. Proposed by HARRY S. VANDIVER, Bala, Pa.

Prove that it is impossible to find integral values for x, y, and z such that the relation $x^2y+xz^2=y^2z$ is satisfied.

II. Solution by W. F. KING, Ottawa, Canada.

We may assume that x, y, and z have no factor common to them all. For if they have a common factor n, each term of the equation $x^2y+xz^2=y^2z$ may be divided by n^3, and there will then be left an equation of the same form, in which x, y, and z have no common factor.

Let the greatest common measure of x and y be a; of y and z, b; and of z and x, c. Then we shall have

$$x=cax_1,$$
$$y=aby_1,$$
$$z=bcz_1.$$

Now observe that since b is the greatest common measure of y and z, ay_1 is prime to cz_1; similarly is bz_1 to ax_1, and cx_1 to by_1.

Hence a, b, and c are all prime to one another; so are x_1, y_1, and z_1 to one another. Also a is prime to z_1, b to x_1, and c to y_1. Substituting in the equation the above values of x, y, and z, and dividing by abc, we have

$$a^2c\,x_1^2y_1+c^2b\,z_1^2x_1=b^2a\,y_1^2z_1.$$

Divide by x_1; then $a^2c\,x_1y_1+c^2b\,z_1^2=\dfrac{b^2a\,y_1^2z_1}{x_1}$.

The left hand side is integral, therefore $b^2a\,y_1^2z_1$ is divisible by x_1 But as above shown, x_1 is prime to b, y_1 and z_1. Hence a/x_1 must be an integer.

Again, divide the above equation by a. Then

$$acx_1^2y_1+\frac{c^2bz_1^2x_1}{a}=b^2y_1^2z_1.$$

Hence, as before, $\dfrac{c^2b\,z_1^2x_1}{a}$ is an integer. But a is prime to b, c, and z_1. Therefore, x_1/a is an integer. Now since a/x_1 and x_1/a are both integers, $x_1=a$. Similarly, $y_1=b$, $z_1=c$. Therefore the equation

$$a^2c\,x_1^2y_1+c^2b\,z_1^2x_1=b^2a\,y_1^2z_1$$

becomes $a^4bc+abc^4=ab^4c$, or dividing by abc, $a^3+c^3=b^3$, a known impossible form.